Path probabilities of continuous time random walks

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Path probabilities of continuous time random walks

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Abstract. Employing the path integral formulation of a broad class of anomalous diffusion processes, we derive the exact relations for the path probability densities of these processes. In particular, we obtain a closed analytical solution for the path probability distribution of a Continuous Time Random Walk (CTRW) process. This solution is given in terms of its waiting time distribution and short time propagator of the corresponding random walk as a solution of a Dyson equation. Applying our analytical solution we derive generalized Feynman–Kac formulae.

Keywords: exact results, stochastic particle dynamics (theory), stochastic processes (theory), diffusion

$^3$ Deceased.
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1. Introduction

Despite the success of the standard stochastic models based on Brownian motion and diffusion processes, it has become apparent that many dynamical systems in diverse fields, ranging from biology to physics, cannot aptly be described within this framework [1–3]. Various generalizations of diffusion processes have been considered to account for such an anomalous diffusion. Of particular importance in this context is the Continuous Time Random Walk (CTRW) [4], which is widely applied to model a variety of different systems and now plays a distinguished role in the description of anomalous fluctuations in complex physical, biological, and chemical systems [2]. The CTRW is defined as a proper random walk with an additional random process governing the waiting time between successive jumps. If the waiting time distribution is assumed to be scale-free, the CTRW can be described by the celebrated fractional Fokker–Planck equation (for a review, we refer the reader to [2]). The corresponding Langevin equation can be formulated by an application of the concept of subordination [5, 6].

In addition to the Langevin equations and Fokker–Planck equations, the concept of path integrals plays an important role in the description of stochastic processes [7, 8]. The path integral formulation is based on the specification of a probability measure assigned to each realization of the process. Hence, the path integral formulation encodes the complete statistical information on the process. The basic idea of how to represent CTRWs in terms of path integrals has been recently presented in [9]. A path integral formulation is of particular importance since CTRWs in general are non-Markovian processes and as such are insufficiently described by single-point probability distributions [13]. It can be obtained by the inclusion of an additional stochastic process for the waiting times. This complies with the inclusion of a secondary field, which is quite common in applications of path integrals to quantum and quantum field theories where it corresponds to the inclusion of a further quantum particle. Thus, the CTRW-path integral might also be of interest from a quantum theoretical point of view. It is worth mentioning that, although the considered

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class of anomalous diffusion processes is generally non-Markovian, they are derived from a class of processes which are genuinely Markovian. The non-Markovian character arises from the projection of the process onto the physical degrees of freedom, i.e. by integration over the additional stochastic process. This procedure is similar to the construction of anomalous diffusion processes by subordination [5,6]. Here we employ the path integral representation of CTRWs to obtain closed form expressions for their path probability densities. In particular, we present an analytical solution for the path probability of a CTRW as the solution of a Dyson equation. Such path probability measures are crucial to identify genuine CTRWs, which cannot be distinguished from other anomalous diffusion processes by merely taking single time probability distributions into account. Furthermore, our solution enables the derivation of generalized Feynman–Kac equations, which extend the results previously obtained by Barkai and co-workers [10,11]. These equations can be used to calculate the distributions of functionals of anomalous diffusion paths [10]. A different approach, using the real space renormalization group has been put forward in [12].

The present article is organized as follows. First we introduce the path integral formulation for a broad class of anomalous diffusion processes based on the ideas presented in [9] and show how CTRWs are contained in this class. Then we derive a closed exact analytical solution for the path probability density of a general CTRW. Finally, we show how the generalized Feynman–Kac formulae can be derived by applying the results from the previous section.

2. Path integral representation

In this manuscript, we consider the processes defined by the one-dimensional discrete Langevin equation:

\[ q_{k+1} = q_k + \tau N(q_k) + \sqrt{\tau D} R_k + \alpha_k r_k, \]  

(1)

where \( k \) is a time index. The first three terms on the right-hand side describe a standard Langevin process with time step \( \tau \), drift \( N(q_k) \) and Gaussian random variables \( R_k \) with amplitude \( \sqrt{D} \). The anomalous contribution stems from the last term, where the \( r_k \) are random variables drawn from a symmetric distribution and \( \alpha_k \) are random variables which we are going to specify later. Observe that the Langevin equation is discretized in such a way that the right-hand side of equation (1) is always taken at the preceding moment of time. This means we work with the Ito regularization, which is the most convenient choice for a functional approach.

Independent of a further specification of the anomalous process the transition amplitude of this process can be obtained as:

\[ p(q_{k+1}|q_k, \alpha_k) = \int \frac{d\tilde{q}_k}{2\pi} e^{i\tilde{q}_k (q_{k+1} - q_k - \tau N(q_k) - \alpha_k r_k) - \frac{\tau^2 \tilde{q}_k^2}{2D}}, \]  

(2)

where we have used the integral representation of the delta-distribution and assumed the stochastic force \( R_k \) to be Gaussian distributed with zero-mean and vanishing correlation \( \langle R_k R_l \rangle = \delta_{kl} \). Since the process we consider is Markovian, the probability for a path
starting at \( q_0 \) is simply given by iteration of equation (2):

\[
g_{N+1}(q_{N+1}; q_N, \alpha_N; \ldots, q_0, \alpha_0) = \int D\tilde{q} e^{S_0(q, \tilde{q})} e^{-\frac{1}{2} \sum_{k=0}^{N} \tilde{q}_k \alpha_k r_k},
\]

(3)

where we have introduced the notation \( D\tilde{q} = \prod_{k=0}^{N} \frac{dq_k}{2\pi} \). The non-anomalous contribution \( S_0(q, \tilde{q}) \) is just the classical action of a diffusion process with drift [14–16] written in the Ito regularized form:

\[
S_0(q, \tilde{q}) = \frac{1}{2} \sum_{k=0}^{N} \tau_{k+1} - q_k - \tau N(q_k) + i \frac{D}{2} \tilde{q}_k^2.
\]

(4)

To further evaluate the expression for the path probability, let us assume that the \( r_k \) are independent Gaussian random variables with zero mean and variance \( \langle r_k r_l \rangle = \tau Q \delta_{kl} \). After averaging with respect to the \( r_k \) we find:

\[
g_{N+1}(q_{N+1}; q_N, \alpha_N; \ldots; q_0, \alpha_0) = \int D\tilde{q} e^{i \sum_{k=0}^{N} \left( \tilde{q}_k (q_{k+1} - q_k - \tau N(q_k)) + \frac{i}{2} (D + Q \alpha_k^2) \tilde{q}_k^2 \right)}
\]

\[
= \frac{1}{\sqrt{\prod_{k=0}^{N} 2\pi (D + Q \alpha_k^2)}} e^{-\frac{1}{D+Q \alpha_k^2} \sum_{k=0}^{N} \left( \frac{(q_{k+1} - q_k - \tau N(q_k))^2}{2(D + Q \alpha_k^2)} + \frac{Q}{2} \tilde{q}_k^2 \right)}.
\]

(5)

This result can be generalized to the case of Lévy-stable random variables. However, to keep the treatment as simple as possible we limit ourselves to the more familiar Gaussian noises. For a path integral formulation of Lévy flight processes we refer to [17].

The probability density for a specific path \( f_{N+1}(q_{N+1}; \ldots; q_0) \) is then given by averaging with respect to the stochastic process \( \alpha_k \):

\[
f_{N+1}(q_{N+1}; \ldots; q_0) = \int D\tilde{q} e^{S_0(q, \tilde{q})} Z \left( i \frac{Q}{2} \tilde{q}_N^2, \ldots, i \frac{Q}{2} \tilde{q}_0^2 \right).
\]

(6)

Here we have introduced the characteristic function:

\[
Z(\eta_N, \ldots, \eta_0) = e^{i \sum_{k=0}^{N} \eta_k \beta_k} = \sum_{\alpha_N} \ldots \sum_{\alpha_0} p(\alpha_N; \ldots; \alpha_0) e^{i \sum_{k=0}^{N} \eta_k \alpha_k^2},
\]

(7)

which is the characteristic function of the process \( \beta_k = \alpha_k^2 \). We have denoted the distribution of \( \alpha = [\alpha_N, \ldots, \alpha_0] \) by \( p(\alpha) \).

How can CTRWs now be described by equation (1)? Let us for a moment set the the drift and the diffusion term to zero, i.e. \( N(q_k) = D = 0 \), and consider the equation:

\[
q_{k+1} = q_k + \alpha_k r_k,
\]

(8)

where the \( r_k \) are Gaussian random variables with zero mean and variance \( \langle r_k r_l \rangle = \tau Q \delta_{kl} \). This equation describes a random walk but with increments whose amplitude is determined by the process \( \alpha \). To see how a CTRW can be described by equation (8) we remind that a CTRW is defined as a random walk where the times between successive relocations are independent, identically distributed (\( iid \)) random variables. In other words, a CTRW can be considered as a random walk which is subordinated to a renewal process governed by some waiting time distribution \( W(\tau) \). To obtain CTRW-dynamics from equation (8) we assume the \( \alpha \)-process to be a binary string consisting only of zeros and ones. Furthermore, we let the number \( \tau \) of consecutive zeros be a random variable drawn from some distribution \( W(\tau) \). This means the \( \alpha \)-process, defined by the distribution
of zeros between two successive ones, is a renewal process. Inserting this process into equation (8), it can easily be seen that this provides a description of CTRW in discrete time, where the random walker stays at its respective position when \( \alpha_k = 0 \), and performs a jump when \( \alpha_k = 1 \). Since the number of zeros between the successive ones (i.e. the jumps), are iid random variables, we obtain CTRW-dynamics. More precisely, equation (8) describes in this case an unbiased uncoupled CTRW with Gaussian increments. Biased CTRWs with drift \( L(q_k) \) can be included by adding a term \( \alpha_k L(q_k) \) to equation (8). The Langevin equation with non-zero drift and diffusion describes then a CTRW with internal dynamics where the random walker is subject to some Markovian dynamics during the waiting time. This class of processes and its applications has been discussed in [18, 19].

### 3. Path probabilities for CTRWs

The starting point of our treatment of CTRW path integrals are equations (6) and (7), and we need to characterize the process \( \alpha \). For the sake of simplicity, we restrict ourselves to processes with \( \alpha_0 = 1 \), i.e. processes that start with an event. The aim is to represent the probability of a specific sequence \( p(\alpha) \) in terms of the waiting time distribution, i.e. the number of zeros between two successive ones. Let \( W_{i,j} \) denote the probability to have \( i - j - 1 \) zeros between ones at \( j \) and \( i, i > j \), and furthermore let \( W_{i,i} = 0 \). It follows that the survival probability \( w_{i,j} \), i.e. the probability that a one at \( j \) is not followed by a further one, until \( i \) is \( w_{i,j} = 1 - \sum_{l=j}^{i} W_{l,j} \).

To proceed it is convenient to introduce the probability \( \nu_{k,0}(1, \alpha_{k-1}, \ldots, \alpha_1, 1) \) of truncated strings, \([1, 0, 0, \ldots, 1, \ldots, 0, 1]\), which end with an event at \( k \) and temporarily also introduce a second index to indicate the starting event. According to the definition of renewal processes the truncated densities fulfill the relation:

\[
\nu_{k,0}(1, \alpha_{k-1}, \ldots, \alpha_1, 1) = \delta_{k,0} + \sum_{l=0}^{k} \sigma_{k,l} \nu_{l,0}(1, \alpha_{l-1}, \ldots, \alpha_1, 1), \tag{9}
\]

where we have defined:

\[
\sigma_{k,l} = \delta_{\alpha_k,1} \delta_{\alpha_{k-1},0} \ldots \delta_{\alpha_{l+1},0} \delta_{\alpha_l,1} W_{k,l}. \tag{10}
\]

The density of the truncated strings of length \( k \) depends on the densities of truncated strings of length \( l < k \). Equation (9) has the form of a Dyson equation [20] and iterative application of (9) allows us to present the truncated density for \( \nu_{k,0} \) in terms of a Dyson series [21]:

\[
\nu_{k,0}(1, \alpha_{k-1}, \ldots, \alpha_1, 1) = \delta_{k,0} + \sum_{l=0}^{k} \sigma_{k,l} \delta_{l,0} + \sum_{l=0}^{k} \sum_{l'=0}^{l} \sigma_{k,l} \sigma_{l',l'} \delta_{l',0} + \ldots \tag{11}
\]

where \( k > l > l' > \ldots > 0 \). To see how this series can be summed, let us write equation (9) in matrix form:

\[
\nu = \delta + \sigma \nu, \tag{12}
\]

where the matrices \( \nu, \delta, \sigma \) have the elements \( \nu_{i,j}, \delta_{i,j}, \sigma_{i,j} \) respectively. Solving for \( \nu \), we obtain for the elements:

\[
\nu_{k,0}(1, \alpha_{k-1}, \ldots, \alpha_1, 1) = \sum_{l=0}^{k} \left[ E - \sigma_{k,l}(\text{E})^{-1} \right] \delta_{l,0}, \tag{13}
\]

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where $E$ denotes the unit matrix.

The probability density $p_{N,0}(\alpha_N, \ldots, \alpha_0)$ of the renewal process is then determined from the truncated densities on the basis of the relationship:

$$p_{N,0}(\alpha_N, \ldots, \alpha_0) = \sum_{k=0}^{N} \gamma_{N,k} \nu_{k,0}(1, \alpha_{l-1}, \ldots, \alpha_1, 1), \quad (14)$$

where we have defined:

$$\gamma_{N,k} = \delta_{\alpha_N,1} \delta_{N,k} + w_{N,k} \delta_{\alpha_N,0} \delta_{\alpha_{N-1},0} \cdots \delta_{\alpha_{k+1},0} \delta_{\alpha_{k+1},1}. \quad (15)$$

The explicit representation of the probability density is then just:

$$p_{N,0}(\alpha_N, \ldots, \alpha_0) = \sum_{k=0}^{N} \sum_{l=0}^{k} \gamma_{N,k} [E - \sigma_l]^{-1} \delta_{l,0}. \quad (16)$$

Again, an expansion of the matrix $[E - \sigma]^{-1}$ yields a representation of the probability density of a string $[0, \ldots, 1, \ldots, 1]$ in the number of ones contained in the string. Concluding, by using equation (7) the characteristic function of the renewal process $\alpha$ can be easily assessed on basis of equation (16):

$$Z(\eta_N, \ldots, \eta_0) = \sum_{k=0}^{N} \sum_{l=0}^{N} \tilde{\gamma}_{N,k} [E - \tilde{\sigma}_l]^{-1} \delta_{l,0}, \quad (17)$$

where now:

$$\tilde{\gamma}_{k,l} = e^{i \eta_k} w_{k,l} e^{i \eta_l}, \quad \tilde{\sigma}_{k,l} = e^{i \eta_k} W_{k,l} e^{i \eta_l}. \quad (18)$$

The combination of equation (7) with equation (17) provides the desired path-integral representation for the class of CTRWs under consideration.

Since this path-integral representation is a rather condensed representation of the CTRW, we proceed to derive a more transparent formulation, which is based on equations (9) and (16). It is clear that the probability distribution $f_{N+1}(q_{N+1}, q_N, \ldots, q_0)$ should have an analogous representation, where we, for the sake of simplicity, from now on omit the starting index again. We introduce the abbreviations $K_0(q_N; \ldots; q_l)$ for the path probability of the classical action equation (4) and $V(q_{l+1}; q_l)$ for the short time propagator of the action with $\alpha_l = 1$:

$$V(q_{l+1}; q_l) = \frac{1}{\sqrt{2\pi \tau(D + Q)}} e^{-\frac{(q_{l+1} - q_l - N(q_l))^2}{2\tau(D + Q)}}. \quad (19)$$

The starting point is the definition of the truncated distribution:

$$\xi_{k+1}(q_{k+1}, q_k, \ldots, q_0) = \sum_{\alpha} \nu_{k}(1, \alpha_{k-1}, \ldots, \alpha_1, 1) \times g_{k+1}(q_{k+1}, q_k, 1; q_{k-1}, \alpha_{k-1}; \ldots; q_0, 1). \quad (20)$$

Using the relation (9) we arrive at a Dyson equation for the truncated distribution:

$$\xi_{k+1}(q_{k+1}, q_k, \ldots, q_0) = \delta_{k,0} V(q_1, q_0) + \sum_{l=0}^{k} V(q_{k+1}, q_k) W_{k,l} K_0(q_k, \ldots, q_{l+1}) \xi_{l+1}(q_{l+1}, \ldots, q_0). \quad (21)$$

Introducing the matrix $\rho$ with the elements:

$$\rho_{k,l} = V(q_{k+1}, q_k) W_{k,l} K_0(q_k, \ldots, q_{l+1}), \quad (22)$$

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we can solve the Dyson equation (21) similarly to the solution of equation (9):

\[ \xi_{k+1}(q_{k+1}, q_k, \ldots, q_0) = \sum_{l=0}^{k} (E - \rho)_{k,l}^{-1} V(q_{l+1}, q_l) \delta_{l,0}. \] (23)

Employing equation (14) we can establish the connection between the distributions \( \xi_{k+1} \) and \( f_{N+1} \):

\[ f_{N+1}(q_{N+1}, q_N, \ldots, q_0) = \sum_{l=0}^{N} w_{N,l} K_0(q_{N+1}, \ldots, q_{l+1}) \xi_{l+1}(q_{l+1}, \ldots, q_0). \] (24)

Combining equations (21) and (24) we finally obtain a closed representation of the joint probability distribution of a CTRW path as:

\[ f_{N+1}(q_{N+1}, q_N, \ldots, q_0) = \sum_{k=0}^{N} w_{N,k} K_0(q_{N+1}, \ldots, q_{k+1}) \sum_{l=0}^{k} (E - \rho)_{k,l}^{-1} V(q_{l+1}, q_l) \delta_{l,0}. \] (25)

The expansion of the inverse matrix then yields an expansion of the path probability in terms of the number of events. The first term \( f^{(0)} \) yields the contribution of paths during which no event occurred after the first one:

\[ f^{(0)}_{N+1}(q_{N+1}; \ldots; q_0) = w_{N,0} K_0(q_{N+1}; \ldots; q_1) V(q_1; q_0), \] (26)

whereas the first order term contains the contribution from paths with one event:

\[ f^{(1)}_{N+1}(q_{n+1}; \ldots; q_0) = \sum_{l=1}^{N} w_{N,l} K_0(q_{N+1}; \ldots; q_{l+1}) V(q_{l+1}; q_l) W_{l,0} K_0(q_l; \ldots; q_1) V(q_1; q_0) \] . (27)

Higher order terms are obtained in the same way. Note the formal analogy to the self-energy corrections in quantum theory. Here, the ‘corrections’ to the free propagator are given by the anomalous contributions from the \( \alpha \)-process. By considering processes which start with an event, we imply non-equilibrium initial conditions. Relaxing this condition would lead to the case of aging CTRWs [22].

To conclude this section, let us show how path probabilities of standard random walks are recovered from our formalism. This case is included for \( N(q_k) = D(q_k) = 0 \) and \( W_{i,j} = \delta_{i,i-1} \), i.e. all \( \alpha_k = 1 \). Defining the matrix \( \rho^{(RW)} \) with the elements \( \rho^{(RW)}_{i,j} = \delta_{i+1,i} V(q_{i+1}; q_i) \) and using equation (25) we obtain:

\[ f_{N+1}(q_{N+1}; \ldots; q_0) = (E - \rho^{(RW)})^{-1}_{N,0} \]
\[ = \prod_{i=0}^{N} V(q_{i+1}; q_i) \]
\[ = (2\pi\tau Q)^{-\frac{N+1}{2}} e^{-\sum_{i=0}^{N} \frac{(q_{i+1} - q_i)^2}{2\pi\tau Q}}, \] (28)

which is a well-known solution for the path probability of a random walk [21].

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4. Generalized Feynman–Kac formula

Closely related to the path probabilities of a stochastic process is the Feynman–Kac equation, which describes the functionals of these processes [23]. Thus, having established the framework to calculate the path probabilities of CTRWs, it is natural to apply the results and to derive generalized Feynman–Kac-equations. To this end let us consider the functional:

$$P_{N+1}(p, q_{N+1}) = \int \prod_{j=0}^{N} dq_j e^{ip \sum_{k=0}^{N} U(q_k)} f_{N+1}(q_{N+1}, \ldots, q_0).$$  \hfill (29)

The quantity $P_{N+1}(0, q_{N+1})$ is just the probability distribution $f_{N+1}(q_{N+1})$, which obeys a generalized Fokker–Planck equation. Defining now:

$$G(q_{N+1}, q_k; p, N, k) = \int \prod_{j=k+1}^{N} dq_j e^{ip \sum_{l=0}^{N} U(q_l)} f_{N+1-k}(q_{N+1}, \ldots, q_k),$$  \hfill (30)

as well as:

$$\zeta_{k+1}(p, q_k) = \int \prod_{j=0}^{k+1} dq_j e^{ip \sum_{l=0}^{j} U(q_l)} \xi_{k+1}(q_{k+1}, \ldots, q_0),$$  \hfill (31)

we obtain from equations (21) and (24) the relations:

$$P_{N+1}(p, q_{N+1}) = \sum_{k=0}^{N} w_{N,k} \int dq_{k+1} G(q_{N+1}, q_{k+1}; p, N, k) \zeta_{k+1}(p, q_{k+1}),$$  \hfill (32)

and:

$$\zeta_{k+1}(p, q_{k+1}) = V(q_1, q_0) \delta_{k,0} + \sum_{l=0}^{k} \int dq_k \int dq_{l+1} V(q_{k+1}, q_k) W_{k,l} G(q_k, q_{l+1}; p, k, l + 1) \times \zeta_{l+1}(p, q_{l+1}).$$  \hfill (33)

Now equations (32) and (33) are just the time-discrete versions of the equations which served as the starting point for the derivation of generalized Fokker–Planck equation for the CTRW with internal dynamics [18], eventually leading to:

$$\left[ \frac{\partial}{\partial t} - \mathcal{H} \right] P(p, q, t) = \int_0^t dt' Q(t - t') \mathcal{L}_1 e^{(t-t') \mathcal{H}} P(p, q, t'),$$  \hfill (34)

where $\mathcal{H} = \mathcal{L}_0 + ip U(q)$. Here $\mathcal{L}_0$ is the generator, i.e. the Fokker–Planck operator, of the stochastic process equation (1) with $\alpha = 0$, $\mathcal{L}_1$ is the generator of the same process with $\alpha = 1$, $Q(t - t')$ is the common time-evolution kernel of CTRWs, whose Laplace transform $\hat{Q}(\lambda)$ is given by $\hat{Q}(\lambda) = \lambda \hat{W}(\lambda)/(1 - \hat{W}(\lambda))$, where $\hat{W}(\lambda)$ is the Laplace transform of the the waiting time distribution, see [2]. For power-law distributed waiting times with a diverging mean, i.e. $W(\tau) ~ \tau^{-1-\delta}$ with $0 \leq \delta \leq 1$, a non-trivial regularization scheme can be employed [19], to obtain a fractional version of equation (34):

$$\left[ \frac{\partial}{\partial t} - \mathcal{H} \right] P(p, q, t) = \mathcal{L}_1 D_t^{1-\delta} P(p, q, t),$$  \hfill (35)

where:

$$D_t^{1-\delta} P(p, q, t) = \frac{1}{\Gamma(\delta)} \left[ \frac{\partial}{\partial t} - \mathcal{H} \right] \int_0^t \frac{dt'}{(t-t')^{1-\delta}} e^{(t-t') \mathcal{H}} P(p, q, t')$$  \hfill (36)

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is the fractional substantial derivative introduced in [19]. For the case of CTRWs without internal dynamics, i.e. $L_0 = 0$, this equation was recently derived in [10] and applications have been presented in [10, 11].

5. Conclusions

Based on their path integral formulation we found a closed form expression for the path probability density of a CTRW in terms of its waiting time distribution. Considering the significance of the CTRW in the stochastic description of many physical, biological, and chemical systems, such path probability measures are crucial since CTRWs cannot be distinguished by just taking single point probability distributions into account. In fact, in the recent article [24] it was shown that two intrinsically different stochastic models, both showing anomalous diffusion, can be governed by the same single-point probability distribution while differing in many other properties such as their autocorrelation functions or their first passage time distributions. Consequently, this work provides a profound motivation for the application of path probabilities in the theory of anomalous diffusion processes since the consideration of single-point probability distributions alone can lead to misleading conclusions. Additionally we have applied our result of the path probabilities to provide an alternative derivation of generalized Feynman–Kac equations, thereby extending the work of Barkai and coworkers to also include the case of CTRWs with internal dynamics.

The calculation of the path probability densities for CTRWs also allows for the first time quantitative predictions about the temporal disorder, i.e. the dynamical randomness, of such systems. These statements can be made by elaborating the relation:

$$f_{N-1}(q_{N-1}, q_{N-2}, ..., q_0) \sim e^{-Nh}$$

(37)

between the path probability density and the entropy production rate. Here the rate $h$ is called the entropy per unit time, which characterizes the temporal disorder of a stochastic process [25].

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