Decay and fragmentation in an open Bose-Hubbard chain

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We analyze the decay of ultracold atoms from an optical lattice with loss from a single lattice site. If the initial state is dynamically stable, a suitable amount of dissipation can stabilize a Bose-Einstein condensate, such that it remains coherent even in the presence of strong interactions. A transition between two dynamical phases is observed if the initial state is dynamically unstable. This transition is analyzed here in detail. For strong interactions, the system relaxes to an entangled quantum state with remarkable statistical properties: The atoms bunch in a few “breathers” forming at random positions. Breathers at different positions are coherent, such that they can be used in precision quantum interferometry and other applications.

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I. INTRODUCTION

Decoherence and dissipation, caused by the irreversible coupling of a quantum system to its environment, represent a major obstacle for a long-time coherent control of quantum states. Sophisticated methods have been developed to maintain coherence also in the presence of dissipation, with applications in quantum control and quantum information processing [1,2]. Only recently a new paradigm has been put forward: Dissipation can be used as a powerful tool to steer the dynamics of complex quantum systems if it can be accurately controlled. It was shown theoretically that dissipative processes can be constructed, which allows one to prepare pure states for quantum computation [3,4], to implement universal quantum computation [5], or to deterministically generate entangled quantum states [6,7].

Ultracold atoms in optical lattices provide a distinguished system to realize new methods of quantum control and quantum-state engineering [8]. Both the coherent dynamics of the atoms and dissipative processes can be accurately controlled, including localized manipulation with single-site resolution [9–14]. In this article we analyze the dynamics induced by the interplay of localized particle dissipation and strong atom-atom interactions. If the interaction strength exceeds a threshold, two metastable equilibria emerge, which can be used to prepare either an almost-pure Bose-Einstein condensate (BEC) or a macroscopically entangled “breather” state.

The metastable breather states show remarkable statistical properties: The atoms relax to a coherent superposition of bunches localized at different lattice positions. Driven by particle loss and interactions, almost all atoms localize in one of the nondissipative wells. The metastable state corresponds to a coherent superposition of these localized modes and thus to a macroscopically entangled quantum state. Because of the tunable large number of atoms forming the breather state, they may serve as a distinguished probe of decoherence and the emergence of classicality. Furthermore, the breather states generalize the so-called NOON states, enabling interferometry beyond the standard quantum limit. As particle loss is an elementary and omnipresent dissipation process, this method may be generalized to a variety of open quantum systems well beyond the dynamics of ultracold atoms, e.g., to optical fiber setups [15] or hybrid quantum systems [16–20].

The paper is organized as follows. After introducing the model system in Sec. II, we analyze breather states in small systems, which allow for a numerically exact simulation of the quantum many-body dynamics in Sec. III. In extended lattices discussed in Sec. V, the localized modes correspond to so-called discrete breathers. The emerging metastable quantum state is more complex, as the atoms can localize in a variety of lattice sites. Nevertheless, one can identify breather states by the number fluctuations and the correlations between neighboring sites. The formation of breather states can be understood to a large extent within a semiclassical phase-space picture introduced in Sec. IV. We analyze the flow of phase-space distribution functions such as the Wigner and the Husimi functions. To leading order it is given by a classical Liouvillian flow which is equivalent to a dissipative Gross-Pitaevskii equation. The emergence of breather states can then be linked to a classical bifurcation of the associated mean-field dynamics. While this semiclassical approach obviously cannot describe the coherence of the quantum state or the formation of entanglement, it correctly predicts the critical interaction strength above which breather states are formed.

II. PARTICLE LOSS IN AN OPTICAL LATTICE

Optical lattices offer unique possibilities in controlling the quantum dynamics of ultracold atoms [8,21]. In particular, experimental parameters such as the strength of the atom-atom interactions can be readily tuned by a variation of the lattice depth. Recently, several experiments demonstrated a local control of the atomic dynamics. Single-site access can be implemented optically either by increasing the lattice period [13,22] or by pushing the resolution of the optical imaging system to the limit [11,12]. Furthermore, the advanced imaging systems in these experiments enable the precise measurement of the atom number per site. An even higher resolution can be realized by a focused electron beam [9,10]. However, the interaction of the electron beam with the atomic cloud is...
generally dissipative: Atoms are ionized and then removed from the lattice by a static electric field. At the same time, this method enables the detection of single atoms with outstanding spatial resolution.

The coherent dynamics of ultracold atoms in deep optical lattices is described by the celebrated Bose-Hubbard Hamiltonian \[ 23 \]

\[ \hat{H} = -J \sum_j (\hat{a}_j^\dagger \hat{a}_{j+1} + \hat{a}_{j+1}^\dagger \hat{a}_j) + \frac{U}{2} \sum_j \hat{a}_j^\dagger \hat{a}_j^\dagger \hat{a}_j \hat{a}_j, \]  

(1)

where \( \hat{a}_j \) and \( \hat{a}_j^\dagger \) are the bosonic annihilation and creation operators in mode \( j \), respectively, \( J \) denotes the tunneling matrix element between the wells, and \( U \) the interaction strength. We set \( \hbar = 1 \), thus measuring the energy in frequency units. This model assumes that the lattice is sufficiently deep, such that the dynamics takes place in the lowest Bloch band only. Throughout this paper we consider finite lattices with \( M \) sites with periodic boundary conditions; i.e., we identify the sites \( j = 0 \) and \( j = M \).

In this article we analyze the nonequilibrium dynamics triggered by localized dissipation implemented by either a resonant laser or a focused electron beam. The atoms are removed rapidly and irreversibly from the lattice, such that the dissipative dynamics can be described by a Markovian master equation,

\[ \frac{d}{dt} \hat{\rho} = -i [\hat{H}, \hat{\rho}] + \mathcal{L} \hat{\rho}. \]  

(2)

Particle loss is described by the Liouvillian \[ 24-28 \]

\[ \mathcal{L}_{\text{loss}} \hat{\rho} = -\frac{1}{2} \sum_j \gamma_j (\hat{a}_j^\dagger \hat{a}_j \hat{\rho} + \hat{\rho} \hat{a}_j^\dagger \hat{a}_j - 2 \hat{a}_j \hat{a}_j^\dagger \hat{\rho}), \]  

(3)

where \( \gamma_j \) denotes the loss rate at site \( j \). Furthermore, the atoms experience phase noise due to collisions with the background gas \[ 29-31 \] or the absorption and spontaneous emission of photons from the lattice beams \[ 32 \]. This dissipation process is described by the Liouvillian

\[ \mathcal{L}_{\text{noise}} \hat{\rho} = -\kappa \sum_j (\hat{\rho} \hat{a}_j^\dagger \hat{a}_j - \hat{a}_j \hat{a}_j^\dagger \hat{\rho} + \hat{\rho} \hat{a}_j \hat{a}_j^\dagger - 2 \hat{a}_j^\dagger \hat{a}_j \hat{\rho}). \]  

(4)

Phase noise can be made very small, e.g., by detuning the optical lattice far from the atomic resonance such that we can assume \( \kappa = 0 \) in most simulations. A detailed analysis of decoherence due to phase noise is then provided in Sec. III D.

For numerical simulations, we make use of the quantum jump method \[ 33,34 \], where the density matrix \( \hat{\rho} \) is decomposed into state vectors,

\[ \hat{\rho} = \frac{1}{L} \sum_{\ell=1}^L |\Psi_\ell\rangle \langle \Psi_\ell| \]  

(5)

whose continuous evolution is interrupted by stochastic quantum jumps. The continuous evolution is determined by the Schrödinger equation with the effective non-Hermitian Hamiltonian,

\[ \hat{H}_{\text{eff}} = \hat{H} - i \frac{1}{2} \sum_j \gamma_j \hat{a}_j^\dagger \hat{a}_j - i \kappa \sum_j \hat{a}_j^\dagger \hat{a}_j^\dagger \]  

(6)

Since \( \hat{H}_{\text{eff}} \) is non-Hermitian, the state vector \( |\Psi\rangle \) must be renormalized after every time step. In the case of particle loss, the state vector jumps according to

\[ |\Psi\rangle \rightarrow \frac{\hat{a}_j^\dagger |\Psi\rangle}{\| \hat{a}_j^\dagger |\Psi\rangle \|}, \]  

(7)

with a probability

\[ \delta \rho = \gamma_j (\langle \hat{a}_j^\dagger \hat{a}_j |\Psi\rangle |\Psi\rangle \delta t \]  

(8)

during a short time interval \( \delta t \). The full density matrix is recovered by averaging over many of these random trajectories in state space.

III. DECAY IN AN OPEN TRIPLE-WELL TRAP

To begin with, we consider the open Bose-Hubbard trimer as an elementary model system, allowing for numerical exact solutions for rather large particle numbers. Still, this model already exhibits the different dynamical phases we aim to understand. A sketch of this system is provided in Fig. 1(a). The bosons tunnel at a rate \( J \) among three lattice sites with periodic boundary conditions. Loss occurs only from the site \( j = 2 \) at a rate \( \gamma_2 \). The system is mirror symmetric with respect to an exchange of sites 1 and 3.

A. Atomic correlations

The most obvious effect of particle dissipation is the decrease in the total particle number \( \hat{n}_{\text{tot}} \) in the lattice, which is shown in the top row in Fig. 2. We simulate the dynamics for three initial states for weak \(( U = 0.01 J )\) and strong \(( U = 0.1 J )\) interactions. A pure BEC with an (anti-)symmetric wave function

\[ |\Psi_\pm\rangle = \frac{1}{\sqrt{2}} (|\hat{a}_1^\dagger \pm \hat{a}_3^\dagger|^N |0\rangle \]  

(9)

and the Fock state

\[ |\Psi_F\rangle = \frac{1}{\sqrt{N/2!}} (|\hat{a}_1^\dagger|^{N/2} |\hat{a}_3^\dagger|^{N/2} |0\rangle, \]  

(10)

assuming that the initial particle number \( N \) is even. The most interesting observation here is that the decay is very slow for the antisymmetric initial state \( |\Psi_-\rangle \). Indeed, this state is a stationary state of the master equation, (2), for \( U = 0 \), such that decay is absent in the noninteracting limit (cf. [27]).

FIG. 1. (Color online) Model systems studied in the present paper: (a) an open Bose-Hubbard trimer with loss from site 2 and periodic boundary conditions; (b) an extended one-dimensional optical lattice with localized loss from a single lattice site.
Density fluctuations and correlations are characterized by the second-order correlation function,

$$g_{j,\ell}^{(2)} = \frac{\langle \hat{a}_j \hat{a}_\ell \rangle}{\langle \hat{a}_j \rangle \langle \hat{a}_\ell \rangle}.$$  \hspace{1cm} (12)

For \( j = \ell \), this expression reduces to the normalized second moment of the number operator \( \langle \hat{n}_j^2 \rangle / \langle \hat{n}_j \rangle^2 \), which quantifies the number fluctuations in the \( j \)th well. The evolution of the number fluctuations and correlations is shown in the bottom panels in Fig. 2. While these quantities are essentially constant for a BEC with a symmetric wave function \( |\Psi_+\rangle \), strong anticorrelations develop for the initial state \( |\Psi_-\rangle \) in the regime of strong interactions. The (anti-)correlations are also found for the Fock state \( |\Psi_F\rangle \), whose experimental preparation can be significantly easier. These results show that the atoms bunch at one of the nondissipative lattice sites, while the other sites are essentially empty. Nevertheless, as we discuss in detail in Sec. III.C, the two contributions localized either at site 1 or at site 3 remain coherent. The atoms thus relax deterministically to a macroscopically entangled state, also called a Schrödinger cat state (cf. [38]). We refer to these states as “breather” states, as they correspond to the so-called discrete breathers in extended lattices in the semiclassical limit [39–44]. This correspondence is discussed in detail in Sec. IV.

B. Transition to the breather regime

The metastable breather state exists only for strong atomic interactions. The onset of breather formation is analyzed in Fig. 3, where we have plotted the total particle number as well as the first- and second-order correlations after a fixed propagation time \( t_{\text{final}} = 50 J^{-1} \) as a function of the interaction strength \( U \). As one can see, for weak interactions, \( U \ll 0.01J \), the BEC remains almost pure and the density-density correlation functions are approximately equal to unity. The characteristic properties of a breather state, strong number fluctuations and anticorrelations between neighboring sites, are observed only for \( U \approx 0.01J \). The transition to the breather regime can be understood within a semiclassical phase-space picture which is discussed in detail in Sec. IV. This approach predicts a bifurcation of metastable states at

FIG. 3. (Color online) Onset of breather formation in a triple-well trap for strong atomic interactions. (a) The total particle number \( n_{\text{tot}} \), (b) the phase coherence \( g_{1,1}^{(1)} \), and (c), (d) the density correlation functions \( g_{1,1}^{(2)} \) and \( g_{1,1}^{(3)} \) as a function of the interaction strength \( U \) after a fixed propagation time \( t_{\text{final}} = 50 J^{-1} \) for \( \gamma_2 = 0.2J \). The initial state is a pure BEC in state \( |\Psi_-\rangle \) with \( N = 60 \) atoms.
a critical interaction strength $U/n_{\text{tot}} = 0.4J$. Before we come back to this issue, we first characterize the quantum properties of the breather states in more detail.

C. Characterization and interferometry of the breather state

In a breather state a large number of atoms localize at a single lattice site, leaving the neighboring sites essentially empty. To make this statement more precise, we analyze the full counting statistics and the coherence of the many-body quantum state in detail. Figures 4(a) and 4(b) show the full counting statistics of the atom number in wells 1 and 2, respectively, at time $t = 10J^{-1}$ after a breather state has formed. The most important result is that the probability distribution $P(n_1)$ becomes bimodal: A breather forms either in the first well ($n_1$ large) or in the third well ($n_1$ almost 0). The second well is almost empty for large values of the interaction constant $U$. This stabilizes the breather state, as only a few atoms are subject to particle loss. For intermediate values of the interaction constant $U$, one also finds the characteristic bimodal number distribution in the first well. However, the atom number in the second well is larger, such that decay is much stronger.

The two breathers at sites 1 and 3 are fully coherent, even for large interactions. To analyze the coherence of the many-body quantum state $\hat{\rho}(t)$ in more detail, we first note that $\hat{\rho}(t)$ can be written as the incoherent sum of contributions with different total particle numbers $n$:

$$\hat{\rho}(t) = \sum_n \rho_n(t) \hat{\rho}^{(n)}(t).$$

There are no coherences between the contributions $\hat{\rho}^{(n)}(t)$, as particle loss proceeds via incoherent jumps only. Numerical results for the density matrix $\hat{\rho}^{(n)}(t)$ with $n = 50$ are shown in Figs. 4(c) and 4(d) at time $t = 10J^{-1}$ after the formation of a breather state. We have plotted the matrix elements $\hat{\rho}^{(n)}_{jk}(t)$ for a subset of matrix indices, fixing $n_2 = n'_2 = 0$ or $n_2 = n'_2 = 1$, respectively. For this plot we simulated the dynamics with the quantum jump method using $L = 3000$ stochastic trajectories in total. One observes that the coherences, i.e., the off-diagonal matrix elements of the projected density matrix, assume their maximum possible values,

$$|\rho_{n_1,n'_1}|^2 \approx |\rho_{n_1,n}| |\rho_{n'_1,n'}|.$$  (14)

This shows that the two breathers formed at lattice sites 1 and 3 are indeed fully coherent. Breather states with different total particle numbers are generally not coherent as discussed above. However, this affects neither the entanglement of the atoms nor its use in quantum interferometry.

Due to the almost-perfect coherence of the modes, breather states enable new applications in precision quantum metrology. In particular, they generalize the so-called NOON states $|n,0,0\rangle + e^{i\theta}|0,0,n\rangle$ which enable precision interferometry beyond the standard quantum limit [20]. Breather states can be written as a superposition of states of the form

$$|n_1,n_2,n-n_1-n_2\rangle + e^{i\theta}|n-n_1-n_2,n_2,n_1\rangle.$$  (15)

That is, the coherence of wells 1 and 3 is guaranteed as in an ordinary NOON state but the total number of atoms forming the NOON state varies statistically. Nevertheless, this is sufficient for precision interferometry.

Starting from the breather state analyzed in the preceding section, we consider an interferometric measurement, where modes (lattice sites) 1 and 3 are mixed. Assuming that interactions (by tuning a Feshbach resonance) and losses are switched off, the dynamics during the interferometer stage is given by the time evolution operator

$$\hat{U}_{\text{interferometer}} = \exp[-i\hat{H}_{\text{mix}}t],$$  (16)

where $\hat{H}_{\text{mix}} = iJ(\hat{a}_1^\dagger\hat{a}_3 - \hat{a}_3^\dagger\hat{a}_1)$. In analogy to the parity observable in NOON state interferometry [19], we record the probability of detecting either an even or an odd number of atoms at site 1. Such a measurement is automatically realized by the optical imaging apparatus in the experiments [11,12].

This probability $P_{\text{even,odd}}$ of detecting an even or an odd number of atoms is plotted in Fig. 5(a) as a function of time. $P_{\text{even}}$ approaches unity periodically at times

$$t_{\text{rev}} = \left(n + \frac{1}{4}\right)\pi J^{-1}, \quad n = 0,1,2,\ldots,$$  (17)

which unambiguously proves the coherence of the breather state. Figures 5(b) and 5(c) show the full counting statistics at site 1 at the beginning of the interferometer stage at $t = 10J^{-1}$ and during the interferometer stage at $t = 13.92J^{-1}$. Destructive interference forbids the detection of an odd number of atoms at this time, such that $P_{\text{even}}(t)$ approaches unity.

The interference fringes observed for $P_{\text{even,odd}}(t)$ are extremely sharp, which enables precision measurement beyond the standard quantum limit. In the present setup, the detection of a fringe reveals the value of the tunneling rate $J$ with ultrahigh precision via Eq. (17). Different quantities can be measured by a modified interferometry scheme as described in [19]. An important but very difficult goal is to increase the number of atoms forming a NOON state (see, e.g., [43]), as...
of multipartite entanglement, we analyze the variance of the overwhelming probability. To unambiguously detect this form remaining atoms will be projected onto the same site with an entangled: If some atoms are measured at one site, then the measurement uncertainty of this method scales inversely with the particle number $n$. This goal may be archived with the breather states discussed here, which are readily generated also for large samples.

D. Entanglement and decoherence

The atoms in a breather or NOON state are strongly entangled: If some atoms are measured at one site, then the remaining atoms will be projected onto the same site with an overwhelming probability. To unambiguously detect this form of multipartite entanglement, we analyze the variance of the population imbalance $\Delta(\hat{n}_1 - \hat{n}_1^2)$, which scales as $-n_1^2$ for a breather state, while it is bounded by $n_{\text{tot}}$ for a pure product state, $n_{\text{tot}}$ being the total atom number. The variance can thus serve as an entanglement criterion, if the quantum state is pure or, more importantly, if one can assure that a large value of the variance is not due to an incoherent mixture of states localized at site 1 or 3.

We assume that a quantum state is decomposed into pure states, $\rho = \frac{1}{L} \sum_{a=1}^{L} |\psi_a\rangle \langle \psi_a|$, as is automatically the case in quantum jump simulations [33]. We then introduce the entanglement parameter,

\begin{equation}
E_{r,q} := \langle (\hat{n}_r - \hat{n}_q)^2 \rangle - \langle \hat{n}_r - \hat{n}_q \rangle^2 - \langle \hat{n}_r + \hat{n}_q \rangle^2 - \frac{1}{2L^2} \sum_{a,b} |\langle \hat{n}_a - \hat{n}_b \rangle_a - \langle \hat{n}_a - \hat{n}_b \rangle_b |^2, \tag{18}
\end{equation}

for wells $(r,q)$, where $\langle \cdot \rangle_{a,b}$ denotes the expectation value in the pure state $|\psi_{a,b}\rangle$. The last term in the parameter $E_{r,q}$ corrects for the possibility of an incoherent superposition of states localized at sites 1 and 3. For a separable quantum state one can now show that $E_{j,k} < 0$ such that a value $E_{j,k} > 0$ unambiguously proves entanglement of the atoms. The detailed derivation is given in Appendix A.

Figure 6 shows the evolution of the entanglement parameter $E_{1,3}(t)$ for three initial states. The symmetric state $|\Psi_s\rangle$, remains close to a pure BEC, such that $E_{1,3}(t) \approx 0$ for all times. In contrast, the antisymmetric state $|\Psi_\pi\rangle$ and the Fock state $|\Psi_F\rangle$ relax to strongly entangled breather states if interactions are sufficiently strong. In this case we observe large positive values of the entanglement parameter $E_{1,3}(t) \approx 1500$ and $E_{1,3}(t) \approx 500$, respectively, which clearly reveals the presence of many-particle entanglement. Notably, entanglement is also generated for the Fock state $|\Psi_F\rangle$ in the regime of weak interactions $\bar{U} = 0.01J$. However, this is only a transient phenomenon caused by interference effects. The breather states formed in the case of strong interactions are metastable such that the generated entanglement persists for long times until all atoms decay from the trap. Thus, localized particle dissipation enables the robust, deterministic generation of entanglement only in the presence of strong interactions.

Furthermore, entangled breather states provide a sensitive probe for environmentally induced decoherence. Figure 7(a) shows the evolution of the entanglement parameter $E_{1,3}(t)$ for three values of the strength of phase noise $\kappa$ starting from the antisymmetric initial state $|\Psi_\pi\rangle$. Entanglement is generated in all cases, but $E_{1,3}(t)$ rapidly decreases again when $\kappa$ is large due to the decoherence of the breathers. Notably, one finds strong number fluctuations $\delta n_{1,3}^{(2)} > 1$ and anticorrelations $g_{1,3}^{(2)} < 1$ also in the presence of strong phase noise, but interferometry is no longer possible. Figure 7(b) shows the maximum value of $E_{1,3}(t)$ realized in the presence of phase noise. Entanglement decreases with the noise rate $\kappa$, in which breather states with large particle numbers are most sensitive. However, entanglement persists up to relatively large values, $\kappa \approx 10^{-2}J$ in all cases.

FIG. 5. (Color online) Interferometry of the NOON state according to the time evolution $U(t) = \exp[-J(\hat{a}_1^\dagger \hat{a}_3 - \hat{a}_3^\dagger \hat{a}_1)t]$. (a) Probability of detecting an even (solid line) and an odd (dashed line) number of atoms at site 1 as a function of time. (b), (c) Full counting statistics at site 1 at (b) the beginning of the interferometer stage $t = 10J^{-1}$ and (c) during the interferometer stage at $t = 13.92J^{-1}$, where $U_{\text{even}} = 1$. The breather state is generated starting from a BEC with an antisymmetric wave function as shown in Fig. 2, with $U = 0.1J$ and $\gamma_2 = 0.2J$. During the interferometer stage we assume that $U = \gamma_2 = 0$.

FIG. 6. (Color online) Evolution of the entanglement parameter, $E_{1,3}$, for three initial states: (a) a BEC with a symmetric wave function $|\Psi_s\rangle$, (b) a BEC with an antisymmetric wave function $|\Psi_\pi\rangle$, and (c) a Fock state $|\Psi_F\rangle$. Parameters are $\gamma_2 = 0.2J$, with $U = 0.01J$ [dashed (blue) line] and $U = 0.1J$ [solid (red) line].
by a classical Liouville equation: discussed in detail in Appendix B. The equations of these distribution functions can be calculated quasi–distribution functions is, to leading order in $1/N$, well and $M$ and $\kappa$ entanglement parameter, $18$, for the antisymmetric initial state \( \psi \). (a) Evolution of the population imbalance between the first and the third sites, $z = (|\alpha_3|^2 - |\alpha_1|^2)/n_0$, vs the relative phase, $\Delta \phi = \phi_3 - \phi_1$. One observes that the trajectory starting at $\Delta \phi = 0$ (red) is dynamically stable, such that it remains in the vicinity of the point $z, \Delta \phi = (0,0)$ for all times. In contrast, trajectories starting close to $z, \Delta \phi = (0, \pi)$ converge to regions with either $z > 0$ or $z < 0$. These regions correspond to self–trapped states, which are known from the nondissipative case [22,48,49]. For $\gamma > 0$, these states become attractively stable, which enables the dynamic formation of breather states. The corresponding quantum dynamics of an initially pure BEC with a (anti-)symmetric $|\psi_{\pm}\rangle$ wave function is shown in Figs. 8(b) and 8(e) and Figs. 8(c) and 8(f), respectively. The Husimi functions of the initial states are localized around $z, \Delta \phi = (0,0)$ and $(z, \Delta \phi = (0, \pi) \; as \; shown \; in \; Figs. \; 8(b) \; and \; 8(c)$. The DGPE then predicts the flow of the Husimi function on a coarse-grained scale. Trajectories starting in the vicinity of $(z, \Delta \phi = (0,0)$ remain close to their initial states, and so does the Husimi function of the symmetric state $|\psi_{-}\rangle$. In contrast, the Husimi function splits up into two fragments localized in the self-trapping regions of phase space for the antisymmetric initial state $|\psi_{+}\rangle$: a breather state is formed. Finally, in Fig. 8(d) the Husimi function of a Fock state is depicted. In this case also the dynamics leads to the split of the function into two parts, as Fig. 8(g) illustrates. However, the number fluctuations and correlations are less pronounced.

The semiclassical picture predicts the fragmentation of the condensate but, of course, cannot assert the coherence and thus the entanglement of the fragments, which is a genuine quantum feature. However, it correctly predicts the stability of an initial state and the emergence of breathers. Thus we can infer the critical interaction strength for the transition to the breather regime from the associated “classical” dynamics. To this end we analyze the metastable states of the DGPE, which are defined as the solutions of the equation

$$-J(\alpha_{\ell-1} + \alpha_{\ell+1}) + U|\alpha_t|^2\alpha_{\ell} - i\frac{\gamma}{2}\delta_{\ell,2}\alpha_{\ell} = (\mu - i\Gamma/2)\alpha_{\ell}.$$ (22)

Here and in the following, we denote by $\tilde{\alpha} = (\alpha_1, \ldots, \alpha_M)^T$ the vector of all amplitudes $\alpha_j$. The metastable states are not stationary states of the DGPE in a strict sense, as the norm and thus the effective nonlinearity $g = U|\tilde{\alpha}|^2$ decay at a rate $\Gamma$. However, if the decay is slow enough and if the solutions are dynamically stable, the time evolution will follow these quasi–steady states adiabatically (cf., e.g., [50]). The properties of the metastable states, their decay rate, and their density distribution are summarized in Fig. 9 as a function of the effective nonlinearity $g$. In the linear case $\gamma = 0$, three solutions exist, which are obtained by a simple diagonalization of the single-particle Hamiltonian. Of particular interest is the antisymmetric state

$$\tilde{\alpha}_{as} = \frac{1}{\sqrt{2}}(1,0,-1),$$ (23)

which exists for all $g$ and has a vanishing decay rate $\Gamma$. With increasing interaction strength $g$, new solutions come into being. At a critical value $g_{cr} = 0.4$, the antisymmetric state $\tilde{\alpha}_{as}$...
bifurcates and two breather solutions emerge. These breathers are strongly localized in one of the nondecaying wells \( j = 1, 3 \). Due to the symmetry of the system, both have the same decay rate \( \Gamma \).

For weak interactions, the state \( \vec{\alpha}_{as} \) dominates the dynamics as its decay rate \( \Gamma \) vanishes. However, this is no longer possible for \( g > g_{cr} \), as these states become dynamically unstable as shown in Fig. 10(a). Instead, the breathers dominate the dynamics. Their decay rate is rather low \([39–42]\), and, most importantly, they are attractively stable as shown in Fig. 10(b). Thus, a breather is formed dynamically during the time evolution for most initial conditions if \( g \) is large enough.

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**FIG. 8.** (Color online) Semiclassical interpretation of breather-state formation. (a) Classical trajectories starting in the vicinity of the symmetric states \((\alpha_1, \alpha_2, \alpha_3) = (1, 0, 1)/\sqrt{2}\) (red) and the antisymmetric states \((1, 0, -1)/\sqrt{2}\) (blue). (b)–(g) The quantum dynamics of the Husimi \( Q \) function follows the classical phase-space trajectories. (b), (e) A BEC with a symmetric wave function \(|\Psi_1\rangle\) remains approximately pure. (c), (f) A BEC with an antisymmetric wave function \(|\Psi_2\rangle\) is coherently split into two parts forming the breather state. (d), (g) A number state \(|\Psi_1F\rangle\) is also split into two parts, but number fluctuations and correlations are less pronounced. The Husimi function \( Q(\alpha_1, \alpha_2, \alpha_3) \) is plotted as a function of the population imbalance \( z = (|\alpha_3|^2 - |\alpha_1|^2)/n_{tot} \) and the relative phase \( \phi_3 - \phi_1 \) for \( \alpha_2 = 0 \) and \( |\alpha_1|^2 + |\alpha_3|^2 = n_{tot} \) at times (b)–(d) \( t = 0 \) and (e)–(g) \( t = 10J^{-1} \). Here, \( n_{tot} \) denotes the total atom number at the respective time. Parameters are \( U = 0.1J, \gamma_2 = 0.2J, \) and \( N(0) = 60 \).

**FIG. 9.** (Color online) Properties of the metastable solutions of the non-Hermitian DGPE, (22), for \( J = 1 \) and \( \gamma_2 = 0.2 \) as a function of the interaction strength \( g = Un_{tot} \). (a) Decay rate per atom \( \Gamma \) and (b) relative occupation of the first well \( |\alpha_1|^2 \). The icons at the right indicate the density distribution in the three wells and the dynamical stability for large \( g \).

**FIG. 10.** (Color online) Analysis of the dynamical stability of the metastable solutions of DGPE (22) for \( J = 1, Un_{tot} = 3, \) and \( \gamma_2 = 0.2 \). The dynamics was simulated starting from a metastable state (black dashed line) and the state plus a small perturbation of order 0.01 (red line). Plotted is the relative occupation of the first well, \( |\alpha_1(t)|^2/||\vec{\alpha}(t)||^2 \). The density distributions of the initial states are illustrated by the icons in the corners: (a) the antisymmetric state, (b) a breather in the first well, (c) a breather in the leaky second well, and (d) the balanced state.
The remaining metastable states are marginally unstable as shown in Figs. 10(c) and 10(d).

V. DECAY IN EXTENDED LATTICES

Next, we are going to discuss how localized single-particle loss affects the dynamics in a more realistic extended lattice. Also, in this case a breather emerges when the interaction strength exceeds a critical value. In the following we analyze the breather formation quantitatively and derive a formula for the critical interaction strength, which depends on the size of the optical lattice. The results presented in this section should be observable in ongoing experiments with ultracold bosons in quasi-one-dimensional optical lattices [9,10,51]. As exact numerical simulations of the many-body quantum dynamics are no longer possible for extended lattices with many atoms, we use the truncated Wigner method (see Appendix B for details). This approximate method is appropriate for a system with large filling factors, since in this case the error is no longer significant for extended lattices with many atoms, we use the truncated Wigner method (see Appendix B for details).

A. Breather-state formation

In the following we consider an extended optical lattice consisting of \( M = 50 \) sites with periodic boundary conditions unless stated otherwise. Loss occurs from the lattice site \( j = 1 \) only. As an initial state we assume a pure BEC which is moved at constant speed [52] or accelerated [53] to the edge of the lattice site \( j = 1 \). This approximate method is appropriate for a system with large filling factors, since in this case the error is no longer significant for extended lattices with many atoms, we use the truncated Wigner method (see Appendix B for details).

The phase coherence \( g^{(1)}_{j,k} \) between adjacent wells is lost after a short transient period, indicating the dynamical instability of the condensate. At the same time the number fluctuations \( g^{(2)}_{j,j+k} \) rapidly increase as shown in Fig. 11(b). This reveals a strong spatial bunching of the atoms, as expected for a breather state. This feature of the dissipative equilibrium state is in strong contrast to the nondissipative case, where repulsive interactions suppress number fluctuations in thermal equilibrium. Figure 11(d) reveals the second characteristic trait of the breather state. Strong anticorrelations with \( g^{(2)}_{j,j+2} \approx 0.5 \) are observed between site \( j = 25 \) and the next-to-nearest neighbor. No anticorrelations are observed for the direct neighbor, as breathers can extend over more than one site in an extended lattice. We thus conclude that the atoms tend to bunch at one site of the lattice, leaving the neighboring sites essentially empty. This is exactly the signature of the breather state in the extended lattice, which we have discussed above for the trimer case.

The transition to the breather regime for strong interactions is further analyzed in Fig. 12, which shows the first- and second-order coherence functions as a function of the interaction strength for a fixed propagation time \( t_{\text{final}} = 50J^{-1} \) at the reference site \( j = 9 \). For \( U N(0) \lesssim 2.5J \) the phase

![Figure 11](image1.png)

**FIG. 11.** (Color online) Dynamics of a leaky Bose-Hubbard chain with 50 wells. We have plotted (a) the atomic density \( \langle \hat{n}_j(t) \rangle \), (b) the number fluctuations \( g^{(2)}_{j,j}(t) \) at each lattice site, as well as (c) the phase coherence \( g^{(1)}_{j,j+k}(t) \) and (d) the density-density correlations \( g^{(2)}_{j,j+k}(t) \) between site \( j = 25 \) and the neighboring sites \( k = 1 \) [solid (red line)] and \( k = 2 \) [dashed (blue line)]. Parameters are \( U N(0) = 25J \), \( \gamma_1 = 2J \), \( M = 50 \), and \( \rho(t = 0) = N/M = 1000 \).

![Figure 12](image2.png)

**FIG. 12.** (Color online) Transition to the breather regime in an open optical lattice. (a) The phase coherence \( g^{(1)}_{j,j+k} \) and (b) the number correlation function \( g^{(2)}_{j,j+k} \) between site \( j = 9 \) and the neighboring sites \( k = 1 \) [solid (red line)] and \( k = 2 \) [dashed (blue line)] after a fixed propagation time \( t_{\text{final}} = 50J^{-1} \). One observes a sharp transition when the interaction strength \( U N(0) \) exceeds a critical value of around 2.5J. Parameters are \( \gamma_1 = 2J \), \( M = 50 \), and the atomic density is \( \rho(t = 0) = N/M = 1000 \).
coherence between neighboring sites is preserved, while the atoms decay from the lattice. For stronger interactions, however, phase coherence is lost and the BEC fragments into a breather state. The second-order correlation function $S^{(2)}_{j,j+2}$ reveals the existence of strong anticorrelations for large values of $U$. However, we observe $S^{(2)}_{j,j+2} > 1$ in the vicinity of the transition point. This is a consequence of the localization of the breathers, which becomes tighter with increasing $U$ [40]. Directly above the transition breathers exist but typically extend over several lattice sites, such that we observe positive correlations at this length scale. Moreover, the formation of breathers suppresses the decay from the lattice, that is, the total particle number decreases more slowly. This is due to the strong localization of the breathers, preventing atoms from tunneling to leaky lattice sites.

Note that the coherence functions show the same qualitative behavior if another lattice site is chosen as the reference site instead of $j = 25$ or $j = 9$. The oscillations we observe in Fig. 12(b) and for intermediate values of $U$ are just a manifestation of the temporal oscillations of $g^{(1)}$ and $g^{(2)}$, as shown in Figs. 11(c) and 11(d).

**B. Critical interaction strength**

Breather formation sets in abruptly when the interaction strength exceeds a critical value $U_{\text{crit}}$. Extensive numerical simulations show that the transition point depends on the size of the lattice, i.e., the number of sites $M$, as shown in Fig. 13. As the lattice becomes larger, breather formation is facilitated such that the critical value $U_{\text{crit}}$ decreases rather rapidly. In these simulations, $U_{\text{crit}}$ was determined as follows. After a fixed propagation time we find the values of the density fluctuations $g^{(2)}_{j,j}$ for different interaction strengths $U$ and for various lattice sites $j$. We identified the critical interaction as the maximum interaction strength at which the density fluctuations at all sites $j$ differ from the value in the noninteracting case, $S^{(2)}_{j,j} = 1$, by less than 5%. In all simulations we have used $\gamma = 2J$ and the same initial density, $\rho(t = 0) = N/M = 1000$. In the following we derive a formula for the critical interaction strength, which will also clarify the microscopic origin of breather formation and its connection to the self-trapping effect. Our considerations follow the reasoning presented in [57] for an analogous mean-field system.

As shown in [57], breather formation is a local process, which occurs if the local effective nonlinearity exceeds a critical value $L$, 

$$Un_j/J \leq L,$$

for at least one lattice site $j$. Then the nonlinearity is strong enough to induce self-trapping at the respective lattice site (cf. also [22], [48], [49]). Starting from this local ansatz, the critical interaction strength can be inferred as follows. Breathers are observed if the probability of satisfying condition (25) exceeds a certain threshold value, 

$$\text{prob}(\exists j : n_j > J/L) \geq P_{\text{th}}.$$  

Hennig and Fleischmann [57] furthermore argue that the probability of observing a certain atom number $n_j$ follows a Poissonian distribution in the diffusive regime, such that the cumulative distribution function is given by $\text{prob}(n_j < n_{\text{crit}}) = 1 - e^{-m_{\text{crit}}/N}$. Using this result for a single lattice site, one calculates the probability of finding at least one $n_j \geq n_{\text{crit}} = JL/U$: 

$$\text{prob}(\exists j : n_j > n_{\text{crit}}) = 1 - \text{prob}(n_j < n_{\text{crit}} \forall j) = 1 - (1 - e^{-m_{\text{crit}}/N})^M.$$ 

Substituting this result into Eq. (26) and solving for $U$ then yields the following condition for the onset of breather formation:

$$U \rho \geq U_{\text{crit}} \rho = \frac{-JL}{\ln[1 - (1 - P_{\text{th}})^{1/M}]}$$  

where $\rho = N/M$ is the atomic density.

The analytic prediction, (27), depends on two parameters, $L$ and $P_{\text{th}}$, which are used as fit parameters to model the numeric results. This fit yields an excellent agreement with the numeric results as shown in Fig. 13. We stress that the decrease in $U_{\text{crit}}$ with increasing lattice site cannot be modeled by a simple algebraic or exponential decay. Notably, we obtain significantly smaller values for $U_{\text{crit}}$ than in [57]. This is attributed to the fact that the unstable initial state considered in this paper, a BEC at the band edge, has a higher energy and thus fragments into breathers much more easily.

**VI. CONCLUSIONS AND OUTLOOK**

Recently, there has been great interest in engineering quantum dynamics by using dissipation in various systems [3–7,9,10,14–16,25–27,29,36,37,41,43,46,47,51,57,58]. In this paper, we report the effects of an elementary dissipation mechanism, localized single-particle loss, in a BEC loaded in a deep optical lattice. Particle losses combined with strong interparticle interactions and discrete geometry can deterministically lead to the formation of quantum superpositions of discrete breathers. For a small trimer system we have discussed...
the properties of these “breather states” in detail, including entanglement, decoherence, and possible applications in precision quantum interferometry. A semiclassical interpretation of breather-state formation has revealed the connection to a classical bifurcation of the associated mean-field dynamics. Furthermore, we have studied the dynamical formation of breather states in extended lattices and we have derived a formula which predicts the critical interaction strength at which breathers start to form in lattices with different size. The formation and the properties of these structures could be readily observed in ongoing experiments with ultracold atoms in optical lattices [9–12,14,22].

Nonlinear structures, like bright or dark solitons, are well known in the context of the Gross-Pitaevskii equation (see, for example, [59–61]). However, this equation cannot give us any information about the quantum nature of the problem; these structures are “classical” objects. With the present work we open a new direction: stable nonlinear structures that exhibit purely quantum properties, such as entanglement. These properties cannot be studied anymore with a simple Gross-Pitaevskii equation (or DGPE for discrete systems) and one should go beyond them to support state-of-the-art experiments.

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APPENDIX A: ENTANGLEMENT CRITERION

In this section we provide a detailed derivation of the entanglement criterion based on (18) which is adapted to the NOON states discussed in the present paper. This result generalizes established entanglement criteria in terms of spin squeezing [62] and is derived in a similar way. In contrast to spin-squeezing inequalities, it shows that a state is entangled if the variance defined in (A2) is larger than a certain threshold value.

We assume that the many-body quantum state $\hat{\rho}$ is decomposed into a mixture of pure states,

$$\hat{\rho} = \sum_a p_a \hat{\rho}_a = \sum_a p_a |\psi_a\rangle \langle \psi_a|,$$

where every pure state $\hat{\rho}_a = |\psi_a\rangle \langle \psi_a|$ has a fixed particle number $N_a$. Note that the quantum jump simulation of the dynamics directly provides such a decomposition. We define the entanglement parameter

$$E_{r,q} := (\langle \hat{n}_r - \hat{n}_q \rangle^2 - (\langle \hat{n}_r \rangle - \langle \hat{n}_q \rangle)^2 - (\langle \hat{n}_r \rangle + \langle \hat{n}_q \rangle)^2 \quad (A2)$$

for sites $r$ and $q$. In this expression $\langle \cdot \rangle_{a,b}$ denotes the expectation value in the pure state $|\psi_a, b\rangle$. Now we can prove that $E_{r,q} < 0$ for every separable state such that a value $E_{r,q} > 0$ unambiguously reveals the presence of many-particle entanglement. Note that $E_{r,q}$ provides an entanglement criterion; it is not a quantitative entanglement measure in the strict sense.

To prove this statement we consider an arbitrary separable state and show that $E_{r,q} < 0$ for this class of states. If a pure state $\hat{\rho}_a$ is separable, it can be written as a tensor product of single-particle states,

$$\hat{\rho}_a = \rho_a^{(1)} \otimes \rho_a^{(2)} \otimes \cdots \otimes \rho_a^{(N_a)}.$$

We furthermore introduce the abbreviation

$$\hat{S}^z_{a} := \hat{n}_r - \hat{n}_q.$$

This operator is also written as a symmetrized tensor product of single-particle operators,

$$\hat{S}^z_{a} = \sum_{k=1}^{N_a} \hat{S}^{(k)}_a \otimes \hat{S}^{(k)}_a \otimes \cdots \otimes \hat{S}^{(k)}_a,$$

where the superscript $(k)$ denotes that the single-particle operator $\hat{S}^{(k)}_a$ acts on the $k$th atom. The single-particle operators are given by

$$\hat{S}^{(k)}_a = \langle r | r \rangle \pm |q \rangle \langle q|,$$

where $|r \rangle$ is the quantum state where the particle is localized at site $r$.

For a separable pure state $\hat{\rho}_a$, the expectation values of the population imbalance $\langle \hat{S}^z_{a} \rangle = \text{tr}[\hat{\rho}_a \hat{S}^z_{a}]$ and its square can be expressed as (dropping the subscript $a$ for notational clarity)

$$\langle \hat{S}^z_{a} \rangle = \sum_{k=1}^{N_a} \text{tr}[\rho^{(k)}_a \hat{S}^{(k)}_a],$$

$$\langle \hat{S}^z_{a} \rangle^2 = \sum_{j,k=1}^{N_a} \text{tr}[(\rho^{(j)}_a \otimes \rho^{(k)}_a)(\hat{S}^{(j)}_a \otimes \hat{S}^{(k)}_a)] + \sum_{j=1}^{N_a} \text{tr}[(\rho^{(j)}_a \hat{S}^{(j)}_a)]^2$$

$$= \sum_{j,k=1}^{N_a} \text{tr}[\rho^{(j)}_a \hat{S}^{(j)}_a] \text{tr}[\rho^{(k)}_a \hat{S}^{(k)}_a]$$

$$- \sum_{j=1}^{N_a} \text{tr}[\rho^{(j)}_a \hat{S}^{(j)}_a] \text{tr}[\rho^{(j)}_a \hat{S}^{(j)}_a] + \sum_{j=1}^{N_a} \text{tr}[(\rho^{(j)}_a \hat{S}^{(j)}_a)]^2$$

$$= \langle \hat{S}^z_{a} \rangle^2 + \sum_{j=1}^{N_a} \text{tr}[(\rho^{(j)}_a \hat{S}^{(j)}_a)]^2 - \text{tr}[(\rho^{(j)}_a \hat{S}^{(j)}_a)]^2.$$

Using $\text{tr}[(\rho^{(j)}_a \hat{S}^{(j)}_a)]^2 = \text{tr}[\rho^{(j)}_a \hat{S}^{(j)}_a]$ we thus find that every pure product state $\hat{\rho}_a$ satisfies the condition

$$\langle \hat{S}^z_{a} \rangle < \langle \hat{S}^z_{a} \rangle^2 \leq \langle \hat{S}^z_{a} \rangle + \sum_{a} p_a \langle \hat{S}^z_{a} \rangle^2,$$

where $|\psi_{a,b} \rangle$ is the quantum state where the particle is localized at site $r$. If the total quantum state $\hat{\rho}$ is separable, such that it can be written as a mixture of separable pure states, (A1), the expectation values are given by

$$\langle \hat{S}^z_{a} \rangle = \sum_{a} p_a \langle \hat{S}^z_{a} \rangle^2$$

$$\leq \langle \hat{S}^z_{a} \rangle + \sum_{a} p_a \langle \hat{S}^z_{a} \rangle^2,$$

$$\langle \hat{S}^z_{a} \rangle^2 = \sum_{a,b} p_a p_b \langle \hat{S}^z_{a} \rangle \langle \hat{S}^z_{b} \rangle$$

$$= \sum_{a} p_a \langle \hat{S}^z_{a} \rangle^2 - \frac{1}{2} \sum_{a,b} p_a p_b (\langle \hat{S}^z_{a} \rangle - \langle \hat{S}^z_{b} \rangle)^2. \quad (A9)$$
We thus find that every separable quantum state satisfies the following inequality for the variance of the population imbalance \( \hat{S}_- \):

\[
(\hat{S}_-) - \langle \hat{S}_- \rangle^2 \leq (\hat{S}_+) + \frac{1}{2} \sum_{a,b} p_a p_b (\langle \hat{S}_- \rangle - \langle \hat{S}_- \rangle)^2.
\]  
(A10)

This inequality for separable quantum states can be rewritten as

\[
E_{r,q} < 0
\]  
(A11)
in terms of the entanglement parameter (A2).

APPENDIX B: TRUNCATED WIGNER FUNCTION DYNAMICS

In this Appendix, we explicitly derive the evolution equation for the Wigner function which corresponds to the master equation, (2). To this end we use the following operator correspondences [2]:

\[
\hat{a}_j \hat{\beta} \leftrightarrow \left( \alpha_j + \frac{1}{2} \frac{\partial}{\partial \alpha_j} \right) \mathcal{W},
\]  
(B1)

\[
\hat{\rho} \hat{a}_j \leftrightarrow \left( \alpha_j - \frac{1}{2} \frac{\partial}{\partial \alpha_j} \right) \mathcal{W},
\]  
(B2)

\[
\hat{a}_j \hat{\alpha}_j \leftrightarrow \left( \alpha_j^* - \frac{1}{2} \frac{\partial}{\partial \alpha_j^*} \right) \mathcal{W},
\]  
(B3)

\[
\hat{\rho} \hat{a}_j \leftrightarrow \left( \alpha_j^* + \frac{1}{2} \frac{\partial}{\partial \alpha_j^*} \right) \mathcal{W},
\]  
(B4)

where \( \alpha_j \) are the eigenvalues of the destruction operator:

\[
\hat{a}_j |\alpha_j\rangle = \alpha_j |\alpha_j\rangle, \quad \langle \alpha_j | \hat{a}_j^\dagger \rangle = \alpha_j^* \langle \alpha_j |.
\]  
(B5)

Substituting these correspondences in the master equation, (2), we obtain the following evolution equation for the Wigner function:

\[
\partial_t \mathcal{W} = 2J \sum_{j=1}^{M-1} \text{Im} \left[ \left( \alpha_j - \frac{1}{2} \frac{\partial}{\partial \alpha_j^*} \right) \left( \alpha_j^* + \frac{1}{2} \frac{\partial}{\partial \alpha_j} \right) - \left( \alpha_j^* - \frac{1}{2} \frac{\partial}{\partial \alpha_j} \right) \left( \alpha_j + \frac{1}{2} \frac{\partial}{\partial \alpha_j^*} \right) \right] \mathcal{W}
\]

\[
+ U \sum_{j=1}^{M} \text{Im} \left( \alpha_j - \frac{1}{2} \frac{\partial}{\partial \alpha_j^*} \right)^2 \left( \alpha_j^* + \frac{1}{2} \frac{\partial}{\partial \alpha_j} \right)^2 \mathcal{W} - \frac{1}{2} \sum_{j=1}^{M} \left( \alpha_j^* - \frac{1}{2} \frac{\partial}{\partial \alpha_j} \right) \left( \alpha_j + \frac{1}{2} \frac{\partial}{\partial \alpha_j^*} \right) \mathcal{W}.
\]  
(B6)

As one can easily see, the above equation includes not only first- and second-order derivatives, but also third-order ones arising from the interaction term [the \( U \)-dependent term in the second line of equation (B6)]. These third-order derivatives make the equation quickly unstable, so an approximate method is needed. One technique that is widely used in optical systems is the truncated Wigner method [54,55], which is a good approximation as far as the mode occupation numbers being large. In this approximation one neglects all the terms that include third-order derivatives; thus we have the equation

\[
\partial_t \mathcal{W} = \sum_j \frac{\partial}{\partial x_j} \left[ J(y_{j+1} + y_{j-1})U(y_j - x_j^2y_j - y_j^3 + \frac{y_j}{2}x_j) + \frac{y_j}{2}x_j \right] \mathcal{W}
\]

\[
+ \sum_j \frac{\partial}{\partial y_j} \left[ - J(x_{j+1} + x_{j-1}) - U(x_j - x_jy_j^2 - x_j^3 + \frac{y_j}{2}x_j) + \frac{y_j}{2}x_j \right] \mathcal{W} + \frac{1}{2} \sum_j \frac{y_j}{4} \left( \frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} \right) \mathcal{W},
\]  
(B7)

where \( x_j \) and \( y_j \) are the real and imaginary part of \( \alpha_j \) respectively.

Equation (B7) is a Fokker-Planck equation, thus it can be rewritten in the language of stochastic differential or Langevin equations. To be more precise, consider the Fokker-Planck equation of the form [56]

\[
\partial_t \mathcal{W} = - \sum_j \frac{\partial}{\partial x_j} A_j(z,t) \mathcal{W}
\]

\[
+ \frac{1}{2} \sum_{j,k} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_k} \left[ B_j(z) B_k^T(z,t) \right]_{jk} \mathcal{W},
\]  
(B8)

where the diffusion matrix \( D = BB^T \) is positive definite.

Now, we can write Eq. (B8) as a system of stochastic equations,

\[
\frac{dz}{dt} = A(z,t) + B(z,t)E(t),
\]  
(B9)

where the real noise sources \( E_j(t) \) have zero mean and satisfy \( \langle E_j(t)E_k(t') \rangle = \delta_{jk} \delta(t-t') \). In our case, Eq. (B7) can be rewritten as

\[
\frac{dx_j}{dt} = -J(y_{j+1} + y_{j-1}) - U(y_j - x_j^2y_j - y_j^3) - \frac{y_j}{2}x_j + \sqrt{\frac{y_j}{2}} \xi_j(t),
\]  
(B10)

\[
\frac{dy_j}{dt} = J(x_{j+1} + x_{j-1}) + U(x_j - x_jy_j^2 - x_j^3) - \frac{y_j}{2}x_j + \sqrt{\frac{y_j}{2}} \eta_j(t),
\]  
(B11)
where \( \xi_j(t), \eta_j(t) \) for \( j = 1, \ldots, M \) are \( \delta \) correlated in time with zero mean. Here it must be noted that \( \xi_j(t) \) and \( \eta_j(t) \) are not real noise sources but are included only to recapture the commutation relations of the operators.

As an initial state one uses a product state of the form
\[
|\Psi(t = 0)\rangle = |\psi_1\rangle|\psi_2\rangle \ldots |\psi_M\rangle,
\]
where \( |\psi_j\rangle \) is a Glauber coherent state in the \( j \)th well. This state represents a pure BEC in a grand-canonical framework. The Wigner function of a Glauber coherent state \( |\psi_j\rangle \) is a Gaussian:
\[
W(\alpha_j, \alpha_j^*) = \frac{2}{\pi} \exp\{-|\alpha_j - \psi_j|^2\}.
\]
Thus we can take the initial values for \( \alpha_j = x_j + iy_j \) to be Gaussian random numbers with mean \( \psi_j \). For a BEC in a Bloch state with quasimomentum \( k \), we have
\[
\psi_j = e^{ikj} \frac{\sqrt{N}}{\sqrt{M}}.
\]
In the text we consider a pure BEC accelerated to the edge of the Brillouin zone such that \( k = \pi \).

The truncated Wigner method is used to calculate the evolution of expectation of symmetrized observables as follows. The Wigner function is treated as a probability distribution in phase space. An ensemble of trajectories is sampled according to Eqs. (B10) and (B11). Then one takes the stochastic average over this ensemble,
\[
\langle O_1 \ldots O_k \rangle_{\text{sym}} = \int \prod_{i=1}^{M} d\alpha_i \; O_1 \ldots O_k \; W(\alpha_1, \alpha_1^*, \ldots) = \frac{1}{N_T} \sum_{\ell=1}^{N_T} O_1 \ldots O_k,
\]
where \( O_j \) stands for \( \alpha_j \) or \( \alpha_j^* \), \( N_T \) is the number of trajectories, and the subscript sym reminds us that only expectation values of symmetrized observables can be calculated.

In Fig. 14 we compare the results of the truncated Wigner approximation with the results of the exact quantum jump method for the triple-well trap studied in Sec. III. The simulations show a very good agreement also in the regime of strong interactions. The only small discrepancy is that oscillations of the correlation functions are slightly less pronounced. As the truncated phase-space approximations become more accurate with increasing filling factors [44,45], we expect that the truncated Wigner simulations discussed in Sec. V are reliable both qualitatively and quantitatively.